Chapter 17: Hints and Selected Solutions

Section 17.1 (page 470)

17.1 Here for your convenience is the truth table ♣(P, Q, R):

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>♣(P, Q, R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>T</td>
<td>F</td>
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<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

This can be nicely captured as follows:

\[ \hat{h}(♣(P, Q, R)) = \hat{h}(Q) \text{ if } \hat{h}(P) = \text{true}; \]

\[ \hat{h}(♣(P, Q, R)) = \hat{h}(R) \text{ if } \hat{h}(P) = \text{false} \]

There are other ways of expressing the same thing, though. Just make sure that your definition gives the truth table above.

17.3 Assume that \( h_1 \) and \( h_2 \) are truth assignments that assign the same value to the atomic sentences in \( S \). We are asked to prove that \( \hat{h}_1(S) = \hat{h}_2(S) \). We prove this by induction on wffs.

**Basis:** In this case \( S \) is itself atomic, so the assumption just immediately gives the result.

**Induction Step.** There are several cases to consider, corresponding to the ways of building up propositional wffs. Here is one of the cases. Suppose that we know the result for \( P \) and \( Q \) and want to show that it is true for \( P \lor Q \). Our induction hypothesis insures us that \( \hat{h}_1(P) = \hat{h}_2(P) \) and \( \hat{h}_1(Q) = \hat{h}_2(Q) \). We know that \( \hat{h}_1(P \lor Q) = \text{true} \) if and only if \( \hat{h}_1(P) = \text{true} \) or \( \hat{h}_1(Q) = \text{true} \), or both, by the definition of \( \hat{h}_1 \) (i.e. by the truth table for \( \lor \)), and similarly for \( \hat{h}_2 \). But then it immediately
follows that $\hat{h}_1(P \lor Q) = \hat{h}_1(P \lor Q)$. We leave the other cases to you. Don’t forget $\neg$.

Section 17.2 (page 477)

17.5 We are given that $T$ is the following set of sentences

$$\{\neg(Cube(a) \lor Small(a)), \ Cube(b) \rightarrow Cube(a), \ Small(a) \lor Small(b)\}$$

and are asked to show that this $T$ formally consistent and formally complete.

Formal consistency: If there were a formal proof of $\bot$ from $T$, then $\bot$ would be true in any world that makes $T$ true, by the Soundness Theorem. But no world makes $\bot$ true, so all we need do is to find one world where all the sentences in $T$ are true. Such a world is shown in below. You should submit a different world.

Formal completeness: By Lemma 5, it suffices to show that for each atomic sentence $A$, either $T \vdash T A$ or $T \vdash T \neg A$. There are only four atomic sentences in our language, Cube(a), Cube(b), Small(a), and Small(b). A proof of $\neg Cube(a)$, $\neg Cube(b)$, $\neg Small(a)$, and $Small(b)$ is shown below. We have left out the justifications so that if you decide to borrow our proof you will still need to think through the proof.
17.6 Hint: You should be able to figure this out easily from Proof 17.5, shown in 17.5 above.

17.7 This time, we are given a set $T$ as follows:

\[
\{\neg(Cube(a) \land Small(a)), Cube(b) \rightarrow Cube(a), Small(a) \lor Small(b)\}
\]

We are instructed to go through the atomic sentences in order, adding those that are not decided nor refuted by the theory constructed up to that point. In our case, the atomic sentences are ordered alphabetically by

\[
Cube(a), Cube(b), Small(a), Small(b),
\]

so this is the order we consider them in the procedure. The first sentence, $Cube(a)$, is not decided one way or the other by our theory. (How can you tell?) Thus, we add it to the theory, giving us a new theory

\[
\{\neg(Cube(a) \land Small(a)), Cube(b) \rightarrow Cube(a), Small(a) \lor Small(b), Cube(a)\}
\]
Next we consider the sentence \( \text{Cube}(b) \). Is it decided? If so, fine. If not, toss it into the set. Continue on through all four atomic sentences, then build a world that make the final theory true.

17.8 Hint: Let \( h \) be any assignment making \( T \) true. If \( h \) makes \( A_7 \) true, it would also have to make \( A_8 \) true, and then \( A_9 \), etc.

17.9 The argument

\[
\begin{align*}
\forall x \text{Dodec}(x) & \rightarrow \forall x \text{Large}(x) \\
\forall x \text{Dodec}(x) & \\
\forall x \text{Large}(x)
\end{align*}
\]

has tautological form

\[
\begin{align*}
A & \rightarrow B \\
A & \\
B
\end{align*}
\]

so it is tautologically valid. Hence it is provable in \( \mathcal{F}_T \), by the completeness theorem for propositional logic.

17.12 The argument

\[
\begin{align*}
\forall x (\text{Dodec}(x) \rightarrow \text{Large}(x)) \\
\exists x \text{Dodec}(x) & \\
\exists x \text{Large}(x)
\end{align*}
\]

has the tautological form

\[
\begin{align*}
A & \\
B & \\
C
\end{align*}
\]

so it is not tautologically valid. Hence it is not provable in \( \mathcal{F}_T \), by the Soundness Theorem for \( \mathcal{F}_T \).

If we replace the meaningful predicates by meaningless predicates we obtain:

\[
\begin{align*}
\forall x (P(x) \rightarrow Q(x)) \\
\exists x P(x) & \\
\exists x Q(x)
\end{align*}
\]
This is clearly valid regardless of what P and Q mean, so the original argument is FO-valid. Hence by the Completeness Theorem for first-order logic, the argument is provable in \( \mathcal{F} \).

**Section 17.3 (page 486)**

**17.18** Hint: The first thing we have to do is to make \textit{Between}(a, b, c) true, due to Sentence 1, and \textit{Larger}(b, a) true due to Sentence 6. Next, look at Sentence 5. In order for it to be true, we must make \textit{Cube}(b) true. And so forth. (You do the rest.) This will give us a bunch of atomic sentences that must be true. We have the built one such world, shown below. There are many such worlds and you should turn in a different one.

17.20 1. The Horn sentence

\[ A \land (\neg A \lor B \lor \neg C) \land \neg C \]

could be rewritten in conditional form as

\[ A \land ((A \land C) \rightarrow B) \land (C \rightarrow \bot) \]
17.31 There are many worlds that will make the first five sentences true. One is shown below. If we apply Cut to the first two sentences, we obtain sentence 7. Applying Cut to 3 and 7 yields 8. Continue on in this way on your own.